Name____________________________________ Class______________ Date ____________

5-5 Reteaching

Theorems About Roots of Polynomial Equations

Problem

What are the rational roots of $6x^4 + 29x^3 + 40x^2 + 7x - 12 = 0$?

Step 1 Determine the factors of the constant term and the factors of the leading coefficient.

constant term: 12 factors: ±1, ±2, ±3, ±4, ±6, ±12
leading coefficient: 6 factors: ±1, ±2, ±3, ±6

Step 2 Find all the possible roots by dividing the factors of the constant term by the factors of the leading coefficient. $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{6}, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

Step 3 Substitute each possible root into the polynomial until you find one that causes the polynomial to equal zero. This is one rational root.

Test $-\frac{3}{2}$; $6\left(-\frac{3}{2}\right)^4 + 29\left(-\frac{3}{2}\right)^3 + 40\left(-\frac{3}{2}\right)^2 + 7\left(-\frac{3}{2}\right) - 12 = 0$ $-\frac{3}{2}$ is a rational root.

Step 4 Factor the polynomial by synthetic division using the first rational root as the divisor.

$-\frac{3}{2}$	6	29	40	7	−12
		−9	−30	−15	12
	6	20	10	−8	0

Step 5 If the dividend is a second-degree polynomial, factor to find any additional rational roots. If the dividend does not factor, there are no additional rational roots. If the dividend is greater than a second-degree polynomial, repeat Steps 1–4 until the dividend is a second-degree polynomial.

$-\frac{4}{3}$	6	20	10	−8
		−8	−16	8
	6	12	−6	0

$6x^2 + 12x - 6 = 0$ does not factor. The rational roots of $6x^4 + 29x^3 + 40x^2 + 7x - 12$ are $-\frac{3}{2}$ and $-\frac{4}{3}$.

Exercises

Find all rational roots for $P(x) = 0$.

1. $P(x) = x^3 - x^2 - 8x + 12$

2. $P(x) = x^4 - 49x^2$

3. $P(x) = 2x^3 - 7x^2 - 21x + 54$

4. $P(x) = x^4 - 2x^3 - 3$

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5-5 Reteaching (continued)

Theorems About Roots of Polynomial Equations

Problem

What is a third-degree polynomial function $y = P(x)$ with rational coefficients so that $P(x) = 0$ has roots –4 and $2 \pm 3i$?

Roots: $-4, 2 - 3i, 2 + 3i$	Because $2 - 3i$ is a root, its complex conjugate $2 + 3i$ is also a root.
$(x + 4)[x - (2 - 3i)][x - (2 + 3i)]$	Write the factored form of the polynomial.
$(x + 4)[x^2 - x(2 + 3i) - x(2 - 3i) + (2 - 3i)(2 + 3i)]$	Multiply the factors.
$(x + 4)[x^2 - 2x - 3ix - 2x + 3ix + 4 + 6i - 6i - 9i^2]$	Multiply.
$(x + 4)[x^2 - 4x + 4 - 9i^2]$	Simplify.
$(x + 4)(x^2 - 4x + 13)$	Combine like terms.
$x^3 + 4x^2 - 4x^2 - 16x + 13x + 52$	Multiply.
$x^3 - 3x + 52$	Combine like terms.

A third-degree polynomial function with rational coefficients so that $P(x) = 0$ has roots –4 and $2 \pm 3i$ is $P(x) = x^3 - 3x + 52$.

Exercises

Write a third-degree polynomial function $y = P(x)$ with rational coefficients so that $P(x) = 0$ has the given roots.

5. $1, 2 - i$

6. $5 + 2i, -2$

7. $3, 6 + i$

8. $-4, \sqrt{2}$

9. $2 - \sqrt{3}, -1$

10. $0, 3 - \sqrt{3}$

11. $3i, 7$

12. $2 + \sqrt{5}, 3$

13. $-3, i$

14. $1 - i, 8$

15. $1, 5i$

16. $2, 4 + i$

17. $3, -4i$

18. $0, 2 - i$

19. $-7, 1 - \sqrt{2}$

20. $-4, -\sqrt{7}$

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5-6 Reteaching

The Fundamental Theorem of Algebra

Problem

What are all the complex roots of $x^4 + x^3 - 2x^2 + 4x - 24 = 0$?

Because this is a fourth-degree polynomial, you know it will have four roots.

Step 1 Because the polynomial is already in standard form, you can use the Rational Root Theorem to determine possible rational roots. The possible rational roots are: ±1, ±2, ±3, ±4, ±6, ±8, ±12, ±24.

Step 2 Evaluate the polynomial for each possible root until you find one that causes the polynomial to equal zero. This is a rational root. In this case, one rational root is 2.

Step 3 Use synthetic division with a divisor of 2 to begin factoring the polynomial.

$$\begin{array}{r|rrrrr} 2 & 1 & 1 & -2 & 4 & -24 \\ & & 2 & 6 & 8 & 24 \\ \hline & 1 & 3 & 4 & 12 & 0 \end{array}$$

$x^3 + 3x^2 + 4x + 12 = 0$

Step 4 Repeat Steps 1-3 until you have a polynomial of degree 2 or less.

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 4 & 12 \\ & & -3 & 0 & -12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

$x^2 + 4 = 0$

Step 5 If the dividend is a second-degree polynomial, factor to find any additional roots. If the dividend does not factor easily, use the Quadratic Formula to find the additional roots.

$$\frac{-0 \pm \sqrt{0^2 - 4(1)(4)}}{2(1)} = \frac{\pm\sqrt{-16}}{2} = \frac{\pm 4i}{2} = \pm 2i$$

The four roots of $x^4 + x^3 - 4x^2 + 2x - 24 = 0$ are 2, –3, $2i$, and $-2i$.

Exercises

Find all the complex roots of each polynomial.

1. $x^4 - 8x^3 + 11x^2 + 40x - 80$

2. $4x^4 - x^3 - 12x^2 + 4x - 16$

3. $x^6 + 2x^5 + 7x^4 + 20x^3 - 21x^2 + 18x - 27$

4. $x^3 - 4x^2 + 4x - 16$

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5-6 Reteaching (continued)

The Fundamental Theorem of Algebra

Problem

What are the zeros of $f(x) = x^3 + 4x^2 - x - 10$?

The possible rational roots are ±1, ±2, ±5, ±10.

$$\begin{array}{r|rrrr} 1 & 1 & 4 & -1 & -10 \\ & & 1 & 5 & 4 \\ \hline & 1 & 5 & 4 & -6 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & 4 & -3 & -10 \\ & & -1 & -3 & 6 \\ \hline & 1 & 3 & -6 & -4 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -1 & -10 \\ & & 2 & 12 & 22 \\ \hline & 1 & 6 & 11 & 12 \end{array} \qquad \begin{array}{r|rrrr} -2 & 1 & 4 & -1 & -10 \\ & & -2 & -4 & 10 \\ \hline & 1 & 2 & -5 & 0 \end{array}$$

Use synthetic division to test each possible rational root until you get a remainder of zero.

So −2 is one of the roots.

$$x^3 + 4x^2 - x - 10 = (x+2)(x^2 + 2x - 5)$$

Use the coefficients from synthetic division to obtain the quadratic factor.

Because $X^2 + 2X - 5$ cannot be factored, use the Quadratic Formula to solve $X^2 + 2X - 5 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-5)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{24}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{6}}{2}$$

$$x = -1 \pm \sqrt{6}$$

The polynomial function $f(x) = x^3 + 4x^2 - x - 10$ has one rational zero, -2, and two irrational zeros, $-1 + \sqrt{6}$ and $-1 - \sqrt{6}$.

Exercises

What are the zeros of each function?

5. $f(x) = x^3 - 2x^2 + 4x - 3$

6. $f(x) = x^3 - 3x^2 - 15x + 125$

7. $f(x) = 3x^3 - 2x^2 - 15x + 10$

8. $f(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

9. $f(x) = x^4 - 3x^2 + 2$

10. $f(x) = x^3 - 2x^2 - 17x - 6$