Exponential and logarithm functions

mc-TY-explogfns-2009-1

Exponential functions and logarithm functions are important in both theory and practice. In this unit we look at the graphs of exponential and logarithm functions, and see how they are related.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- specify for which values of a the exponential function $f(x) = a^x$ may be defined,
- recognize the domain and range of an exponential function,
- identify a particular point which is on the graph of every exponential function,
- specify for which values of a the logarithm function $f(x) = \log_a x$ may be defined,
- recognize the domain and range of a logarithm function,
- identify a particular point which is on the graph of every logarithm function,
- understand the relationship between the exponential function $f(x) = \mathrm{e}^x$ and the natural logarithm function $f(x) = \ln x$.

Contents

1.	Exponential functions	2
2.	Logarithm functions	5
3.	The relationship between exponential functions and logarithm functions	9

1. Exponential functions

Consider a function of the form $f(x) = a^x$, where $a > 0$. Such a function is called an *exponential* function. We can take three different cases, where $a = 1$, $0 < a < 1$ and $a > 1$.

If $a = 1$ then

$$f(x) = 1^x = 1.$$

So this just gives us the constant function $f(x) = 1$.

What happens if $a > 1$? To examine this case, take a numerical example. Suppose that $a = 2$.

$$f(x) = 2^x$$

$$f(0) = 2^0 = 1$$

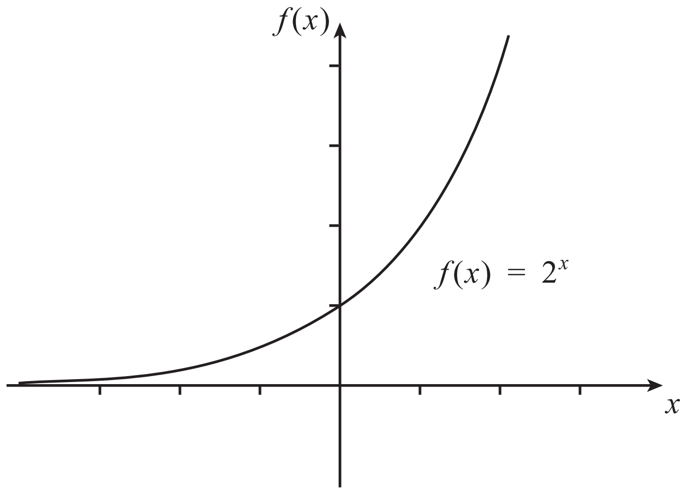
$$f(1) = 2^1 = 2 \qquad f(-1) = 2^{-1} = 1/2^1 = \tfrac{1}{2}$$

$$f(2) = 2^2 = 4 \qquad f(-2) = 2^{-2} = 1/2^2 = \tfrac{1}{4}$$

$$f(3) = 2^3 = 8 \qquad f(-3) = 2^{-3} = 1/2^3 = \tfrac{1}{8}$$

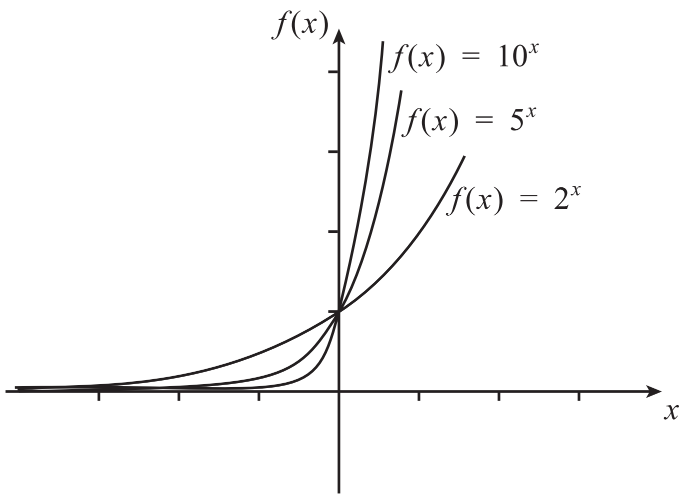
We can put these results into a table, and plot a graph of the function.

x	$f(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



This example demonstrates the general shape for graphs of functions of the form $f(x) = a^x$ when $a > 1$.

What is the effect of varying a? We can see this by looking at sketches of a few graphs of similar functions.



The important properties of the graphs of these types of functions are:

- $f(0) = 1$ for all values of a. This is because $a^0 = 1$ for any value of a.
- $f(x) > 0$ for all values of a. This is because $a > 0$ implies $a^x > 0$.

What happens if $0 < a < 1$? To examine this case, take another numerical example. Suppose that $a = \frac{1}{2}$.

$$f(x) = \left(\tfrac{1}{2}\right)^x$$

$$f(0) = \left(\tfrac{1}{2}\right)^0 = 1$$

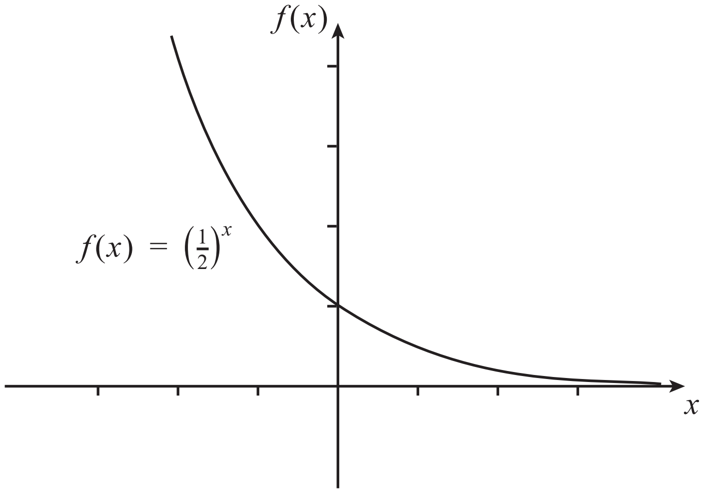
$$f(1) = \left(\tfrac{1}{2}\right)^1 = \left(\tfrac{1}{2}\right) \qquad f(-1) = \left(\tfrac{1}{2}\right)^{-1} = \left(\tfrac{2}{1}\right)^1 = 2$$

$$f(2) = \left(\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{4}\right) \qquad f(-2) = \left(\tfrac{1}{2}\right)^{-2} = \left(\tfrac{2}{1}\right)^2 = 4$$

$$f(3) = \left(\tfrac{1}{2}\right)^3 = \left(\tfrac{1}{8}\right) \qquad f(-3) = \left(\tfrac{1}{2}\right)^{-3} = \left(\tfrac{2}{1}\right)^3 = 8$$

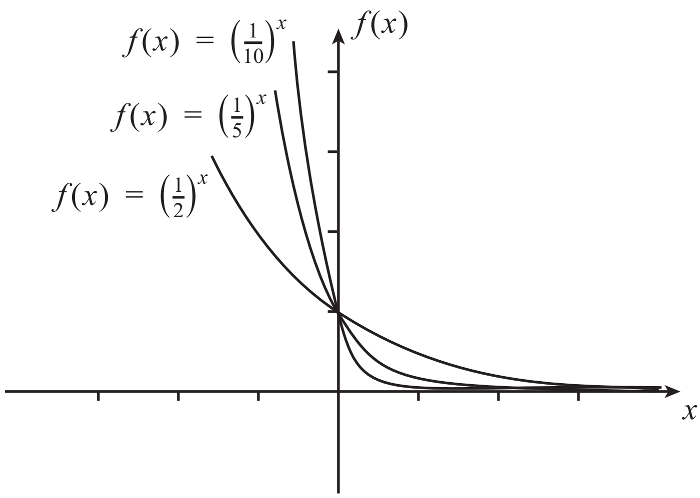
We can put these results into a table, and plot a graph of the function.

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



This example demonstrates the general shape for graphs of functions of the form $f(x) = a^x$ when $0 < a < 1$.

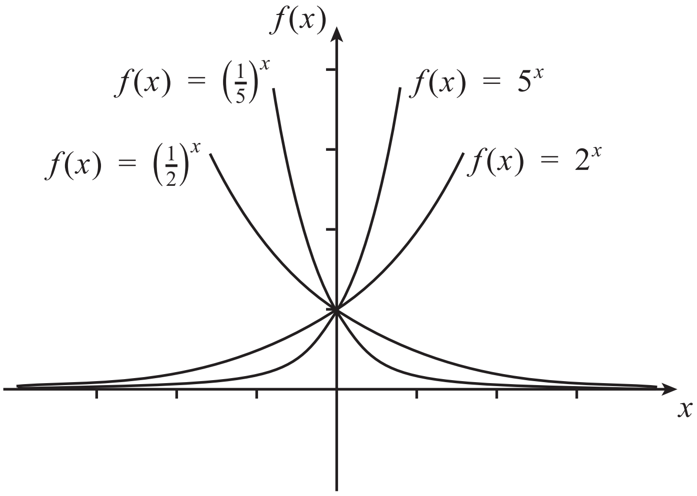
What is the effect of varying a? Again we can see by looking at sketches of a few graphs of similar functions.



The important properties of the graphs of these types of functions are:

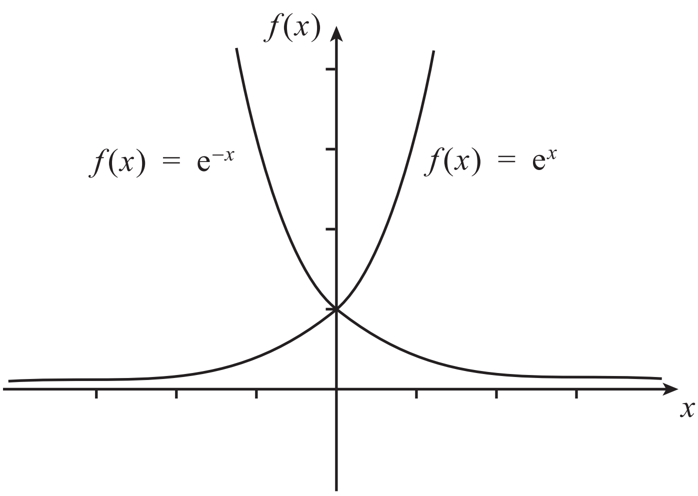
- $f(0) = 1$ for all values of a. This is because $a^0 = 1$ for any value of a.
- $f(x) > 0$ for all values of a. This is because $a > 0$ implies $a^x > 0$.

Notice that these properties are the same as when $a > 1$. One interesting thing that you might have spotted is that $f(x) = (\frac{1}{2})^x = 2^{-x}$ is a reflection of $f(x) = 2^x$ in the $f(x)$ axis, and that $f(x) = (\frac{1}{5})^x = 5^{-x}$ is a reflection of $f(x) = 5^x$ in the $f(x)$ axis.



In general, $f(x) = (1/a)^x = a^{-x}$ is a reflection of $f(x) = a^x$ in the $f(x)$ axis.

A particularly important example of an exponential function arises when $a = \mathrm{e}$. You might recall that the number e is approximately equal to 2.718. The function $f(x) = \mathrm{e}^x$ is often called 'the' exponential function. Since $\mathrm{e} > 1$ and $1/\mathrm{e} < 1$, we can sketch the graphs of the exponential functions $f(x) = e^x$ and $f(x) = e^{-x} = (1/\mathrm{e})^x$.



Key Point

A function of the form $f(x) = a^x$ (where $a > 0$) is called an exponential function.

The function $f(x) = 1^x$ is just the constant function $f(x) = 1$.

The function $f(x) = a^x$ for $a > 1$ has a graph which is close to the x-axis for negative x and increases rapidly for positive x.

The function $f(x) = a^x$ for $0 < a < 1$ has a graph which is close to the x-axis for positive x and increases rapidly for decreasing negative x.

For any value of a, the graph always passes through the point $(0, 1)$. The graph of $f(x) = (1/a)^x = a^{-x}$ is a reflection, in the vertical axis, of the graph of $f(x) = a^x$.

A particularly important exponental function is $f(x) = \mathrm{e}^x$, where $\mathrm{e} = 2.718\ldots$. This is often called 'the' exponential function.

2. Logarithm functions

We shall now look at logarithm functions. These are functions of the form $f(x) = \log_a x$ where $a > 0$. We do not consider the case $a = 1$, as this will not give us a valid function.

What happens if $a > 1$? To examine this case, take a numerical example. Suppose that $a = 2$. Then

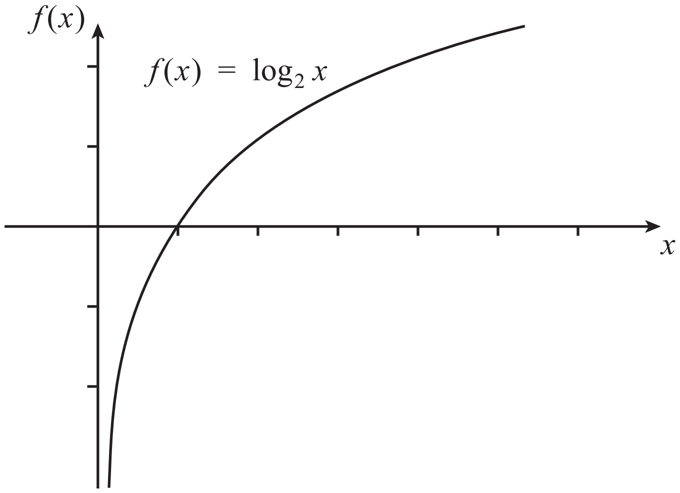
$$f(x) = \log_2 x \qquad \text{means} \qquad 2^{f(x)} = x\,.$$

An important point to note here is that, regardless of the argument, $2^{f(x)} > 0$. So we shall consider only positive arguments.

$$\begin{array}{llll}
f(1) = \log_2 1 & \text{means} & 2^{f(1)} = 1 & \text{so} \quad f(1) = 0 \\
f(2) = \log_2 2 & \text{means} & 2^{f(2)} = 2 & \text{so} \quad f(2) = 1 \\
f(4) = \log_2 4 & \text{means} & 2^{f(4)} = 4 & \text{so} \quad f(4) = 2 \\
f(\frac{1}{2}) = \log_2(\frac{1}{2}) & \text{means} & 2^{f(\frac{1}{2})} = \frac{1}{2} = 2^{-1} & \text{so} \quad f(\frac{1}{2}) = -1 \\
f(\frac{1}{4}) = \log_2(\frac{1}{4}) & \text{means} & 2^{f(\frac{1}{4})} = \frac{1}{4} = 2^{-2} & \text{so} \quad f(\frac{1}{4}) = -2
\end{array}$$

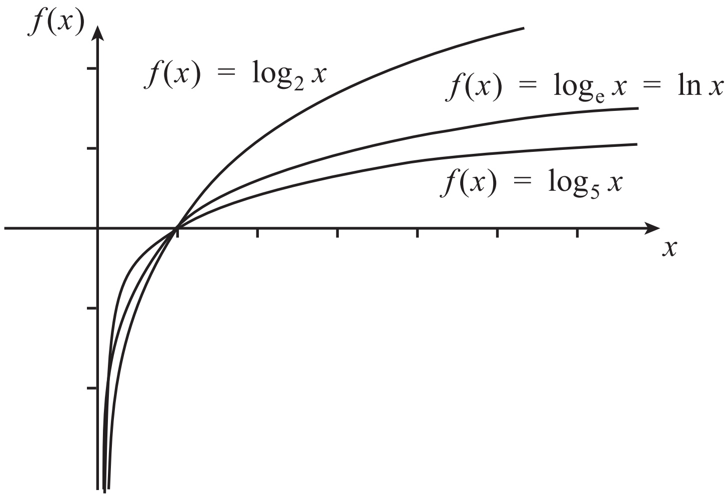
We can put these results into a table, and plot a graph of the function.

x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2



This example demonstrates the general shape for graphs of functions of the form $f(x) = \log_a x$ when $a > 1$.

What is the effect of varying a? We can see by looking at sketches of a few graphs of similar functions. For the special case where $a = \mathrm{e}$, we often write $\ln x$ instead of $\log_\mathrm{e} x$.



The important properties of the graphs of these types of functions are:

- $f(1) = 0$ for all values of a;
- we must have $x > 0$ for all values of a.

What happens if $0 < a < 1$? To examine this case, take another numerical example. Suppose that $a = \frac{1}{2}$. Then

$$f(x) = \log_{1/2} x \qquad \text{means} \qquad \left(\tfrac{1}{2}\right)^{f(x)} = x\,.$$

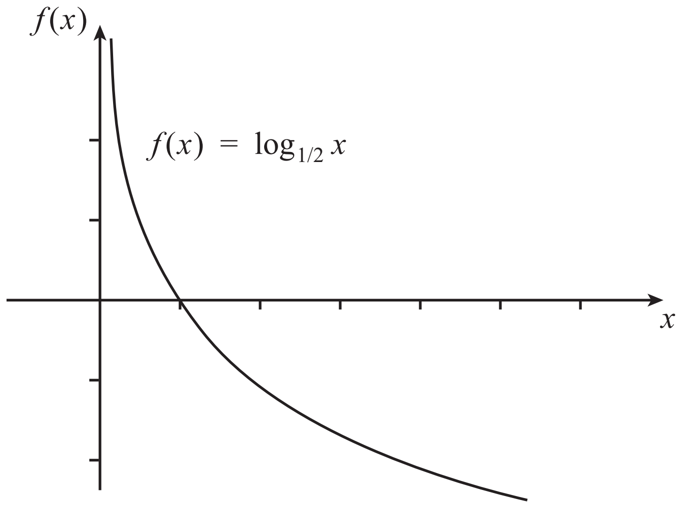
An important point to note here is that, regardless of the argument, $\left(\frac{1}{2}\right)^{f(x)} > 0$. So we shall consider only positive arguments.

$$f(x) = \left(\tfrac{1}{2}\right)^x$$

$$
\begin{array}{llll}
f(1) = \log_{1/2} 1 & \text{means} & \left(\tfrac{1}{2}\right)^{f(1)} = 1 & \text{so} \quad f(1) = 0 \\
f(2) = \log_{1/2} 2 & \text{means} & \left(\tfrac{1}{2}\right)^{f(2)} = 2 = \left(\tfrac{1}{2}\right)^{-1} & \text{so} \quad f(2) = -1 \\
f(4) = \log_{1/2} 4 & \text{means} & \left(\tfrac{1}{2}\right)^{f(4)} = 4 = \left(\tfrac{1}{2}\right)^{-2} & \text{so} \quad f(4) = -2 \\
f(\tfrac{1}{2}) = \log_{1/2}(\tfrac{1}{2}) & \text{means} & \left(\tfrac{1}{2}\right)^{f(\frac{1}{2})} = \tfrac{1}{2} & \text{so} \quad f(\tfrac{1}{2}) = 1 \\
f(\tfrac{1}{4}) = \log_{1/2}(\tfrac{1}{4}) & \text{means} & \left(\tfrac{1}{2}\right)^{f(\frac{1}{4})} = \tfrac{1}{4} = \left(\tfrac{1}{2}\right)^{2} & \text{so} \quad f(\tfrac{1}{4}) = 2
\end{array}
$$

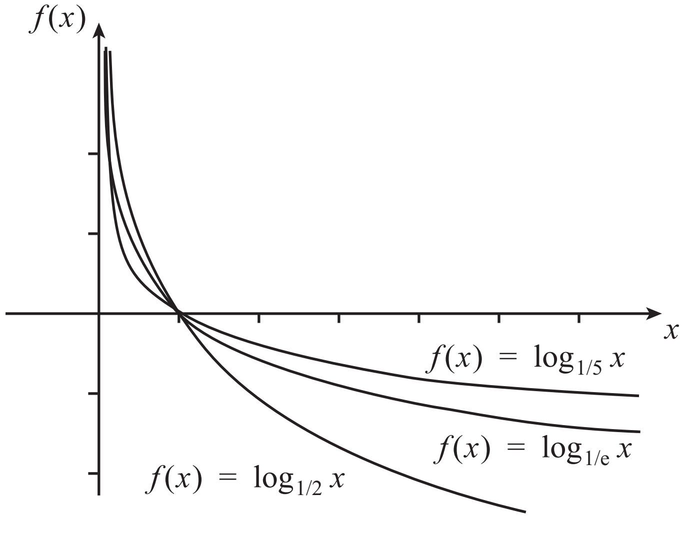
We can put these results into a table, and plot a graph of the function.

x	$f(x)$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2



This example demonstrates the general shape for graphs of functions of the form $f(x) = \log_a x$ when $0 < a < 1$.

What is the effect of varying a? Again we can see by looking at sketches of a few graphs of similar functions.



The important properties of the graphs of these types of functions are:

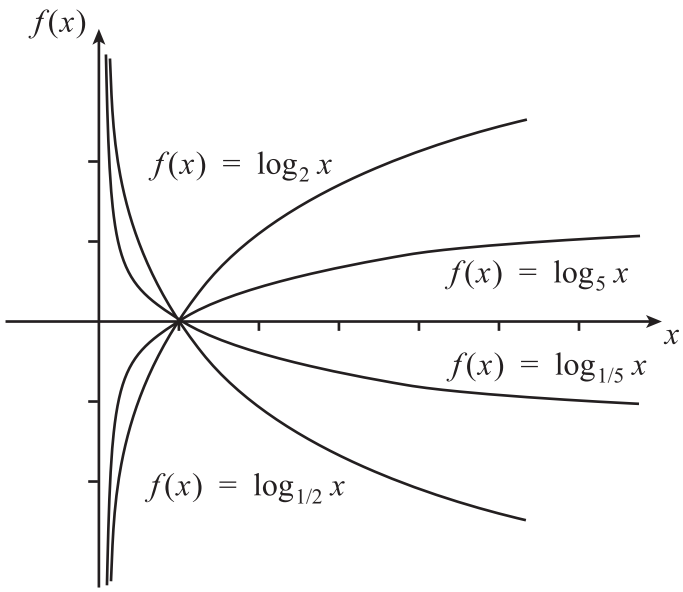
- $f(1) = 0$ for all values of a;
- we must have $x > 0$ for all values of a.

An interesting thing that you might well have spotted is that

$$f(x) = \log_{1/5} x \text{ is a reflection of } f(x) = \log_5 x \text{ in the } x\text{-axis}$$

and

$$f(x) = \log_{1/2} x \text{ is a reflection of } f(x) = \log_2 x \text{ in the } x\text{-axis.}$$



Generally,

$$f(x) = \log_{1/a} x \text{ is a reflection of } f(x) = \log_a x \text{ in the } x\text{-axis.}$$

Key Point

A function of the form $f(x) = \log_a x$ (where $a > 0$ and $a \neq 1$) is called a logarithm function.

The function $f(x) = \log_a x$ for $a > 1$ has a graph which is close to the negative $f(x)$-axis for $x < 1$ and increases slowly for positive x.

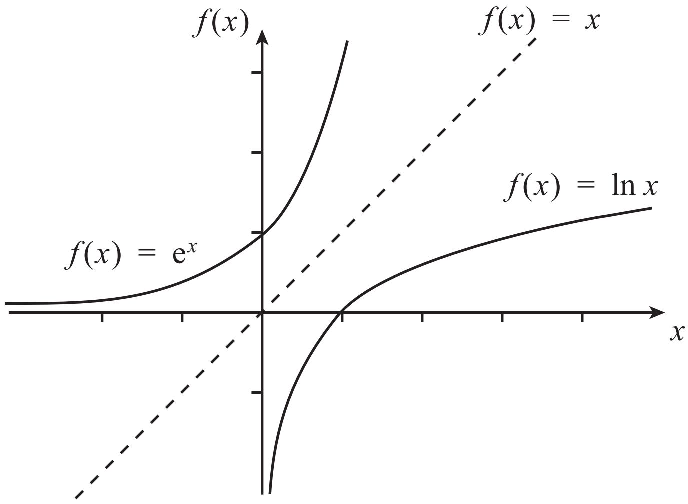
The function $f(x) = \log_a x$ for $0 < a < 1$ has a graph which is close to the positive $f(x)$-axis for $x < 1$ and decreases slowly for positive x.

For any value of a, the graph always passes through the point $(1, 0)$. The graph of $f(x) = \log_{1/a} x$ is a reflection, in the horizontal axis, of the graph of $f(x) = \log_a x$.

A particularly important logarithm function is $f(x) = \log_{\mathrm{e}} x$, where $\mathrm{e} = 2.718\ldots$. This is often called the natural logarithm function, and written $f(x) = \ln x$.

3. The relationship between exponential functions and logarithm functions

We can see the relationship between the exponential function $f(x) = e^x$ and the logarithm function $f(x) = \ln x$ by looking at their graphs.



You can see straight away that the logarithm function is a reflection of the exponential function in the line represented by $f(x) = x$. In other words, the axes have been swapped: x becomes $f(x)$, and $f(x)$ becomes x.

Key Point

The exponential function $f(x) = e^x$ is the inverse of the logarithm function $f(x) = \ln x$.

Exercises

1. Sketch the graph of the function $f(x) = a^x$ for the following values of a, on the same axes.

(a) $a = 3$ (b) $a = 6$ (c) $a = 1$ (d) $a = \frac{1}{3}$ (e) $a = \frac{1}{6}$

2. Sketch the graph of the function $f(x) = \log_a x$ for the following values of a, on the same axes.

(a) $a = 3$ (b) $a = 6$ (c) $a = \frac{1}{3}$ (d) $a = \frac{1}{6}$

3. For each of the following pairs of functions, state whether the graphs are related by a reflection in the x-axis, a reflection in the $f(x)$-axis, a reflection in the line $f(x) = x$, a reflection in the line $f(x) = -x$, or that the graphs are not related by any of these reflections.

(a) $f(x) = 3^x$ and $f(x) = \left(\frac{1}{3}\right)^x$

(b) $f(x) = \log_6 x$ and $f(x) = 6^x$

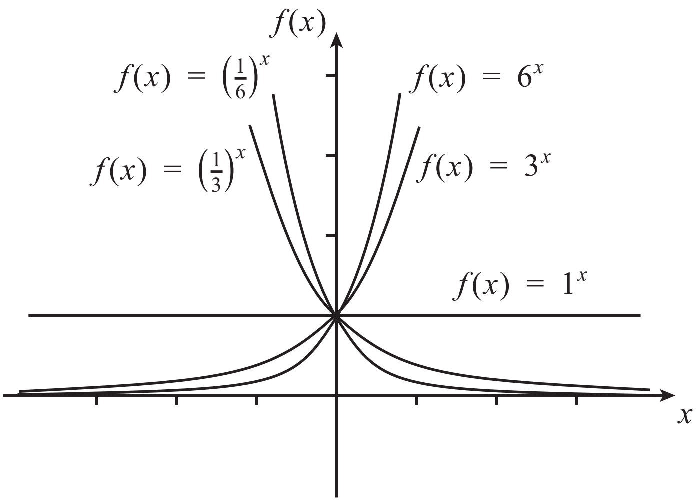
(c) $f(x) = \log_6 x$ and $f(x) = \left(\frac{1}{6}\right)^x$

(d) $f(x) = \log_{1/3} x$ and $f(x) = \log_3 x$

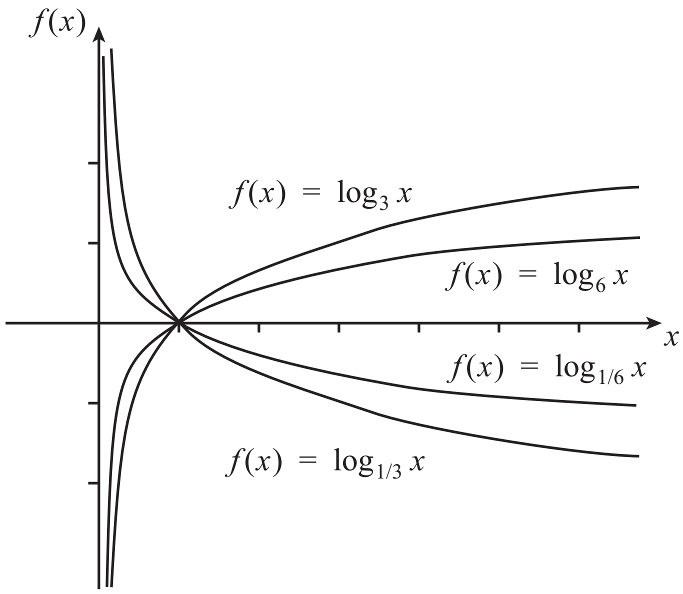
(e) $f(x) = \left(\frac{1}{3}\right)^x$ and $f(x) = \left(\frac{1}{6}\right)^x$

Answers

1.



2.



3.

(a) Reflect in the $f(x)$-axis
(b) Reflect in the line $f(x) = x$
(c) Not related by any of these reflections
(d) Reflect in the x-axis
(e) Not related by any of these reflections

The exponential constant e

mc-bus-expconstant-2009-1

Introduction

The letter e is used in many mathematical calculations to stand for a particular number known as the **exponential constant**. This leaflet provides information about this important constant, and the related **exponential function**.

The exponential constant

The exponential constant is an important mathematical constant and is given the symbol e. Its value is approximately 2.718. It has been found that this value occurs so frequently when mathematics is used to model physical and economic phenomena that it is convenient to write simply e.

It is often necessary to work out powers of this constant, such as e^2, e^3 and so on. Your scientific calculator will be programmed to do this already. You should check that you can use your calculator to do this. Look for a button marked e^x, and check that

$$e^2 = 7.389, \qquad \text{and} \qquad e^3 = 20.086$$

In both cases we have quoted the answer to three decimal places although your calculator will give a more accurate answer than this.

You should also check that you can evaluate negative and fractional powers of e such as

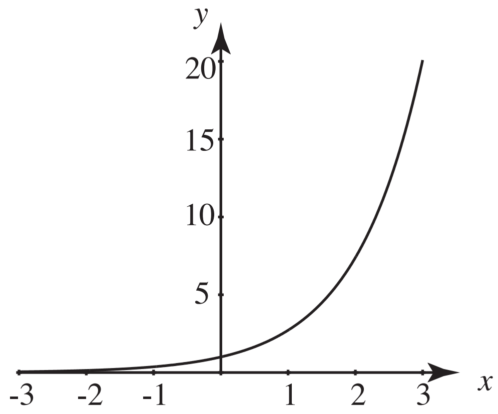
$$e^{1/2} = 1.649 \qquad \text{and} \qquad e^{-2} = 0.135$$

The exponential function

If we write $y = e^x$ we can calculate the value of y as we vary x. Values obtained in this way can be placed in a table. For example:

x	-3	-2	-1	0	1	2	3
$y = e^x$	0.050	0.135	0.368	1	2.718	7.389	20.086

This is a table of values of the **exponential function** e^x. If pairs of x and y values are plotted we obtain a **graph** of the exponential function as shown overleaf. If you have never seen this function before it will be a worthwhile exercise to plot it for yourself.



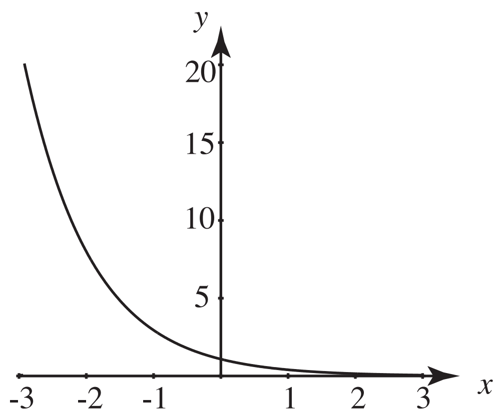
A graph of the exponential function $y = e^x$

It is important to note that as x becomes larger, the value of e^x grows without bound. We write this mathematically as $e^x \to \infty$ as $x \to \infty$. This behaviour is known as **exponential growth**.

The negative exponential function

A related function is the **negative exponential function** $y = e^{-x}$. A table of values of this function is shown below together with its graph.

x	-3	-2	-1	0	1	2	3
$y = e^{-x}$	20.086	7.389	2.718	1	0.368	0.135	0.050



A graph of the negative exponential function $y = e^{-x}$

It is very important to note that as x becomes larger, the value of e^{-x} approaches zero. We write this mathematically as $e^{-x} \to 0$ as $x \to \infty$. This behaviour is known as **exponential decay**.

Exercises

A useful exercise would be to draw up tables of values and plot graphs of some related functions:

a) $y = e^{2x}$, b) $y = e^{0.5x}$, c) $y = -e^x$, d) $y = -e^{-x}$, e) $y = 1 - e^{-x}$.

Logarithms

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Logarithms appear in all sorts of calculations in engineering and science, business and economics. Before the days of calculators they were used to assist in the process of multiplication by replacing the operation of multiplication by addition. Similarly, they enabled the operation of division to be replaced by subtraction. They remain important in other ways, one of which is that they provide the underlying theory of the logarithm function. This has applications in many fields, for example, the decibel scale in acoustics.

In order to master the techniques explained here it is vital that you do plenty of practice exercises so that they become second nature.

After reading this text and / or viewing the video tutorial on this topic you should be able to:

- explain what is meant by a logarithm
- state and use the laws of logarithms
- solve simple equations requiring the use of logarithms.

Contents

1.	Introduction		2
2.	Why do we study logarithms ?		2
3.	What is a logarithm ?	if $x = a^n$ then $\log_a x = n$	3
4.	Exercises		4
5.	The first law of logarithms	$\log_a xy = \log_a x + \log_a y$	4
6.	The second law of logarithms	$\log_a x^m = m \log_a x$	5
7.	The third law of logarithms	$\log_a \frac{x}{y} = \log_a x - \log_a y$	5
8.	The logarithm of 1	$\log_a 1 = 0$	6
9.	Examples		6
10.	Exercises		8
11.	Standard bases 10 and e	$\log$ and $\ln$	8
12.	Using logarithms to solve equations		9
13.	Inverse operations		10
14.	Exercises		11

1. Introduction

In this unit we are going to be looking at logarithms. However, before we can deal with logarithms we need to revise indices. This is because logarithms and indices are closely related, and in order to understand logarithms a good knowledge of indices is required.

We know that

$$16 = 2^4$$

Here, the number 4 is the **power**. Sometimes we call it an **exponent**. Sometimes we call it an **index**. In the expression 2^4, the number 2 is called the **base**.

Example

We know that $64 = 8^2$.

In this example 2 is the power, or exponent, or index. The number 8 is the base.

2. Why do we study logarithms ?

In order to motivate our study of logarithms, consider the following:

we know that $16 = 2^4$. We also know that $8 = 2^3$

Suppose that we wanted to multiply 16 by 8.

One way is to carry out the multiplication directly using long-multiplication and obtain 128. But this could be long and tedious if the numbers were larger than 8 and 16. Can we do this calculation another way using the powers ? Note that

$$16 \times 8 \quad \text{can be written} \quad 2^4 \times 2^3$$

This equals

$$2^7$$

using the rules of indices which tell us to add the powers 4 and 3 to give the new power, 7. What was a multiplication sum has been reduced to an addition sum.

Similarly if we wanted to divide 16 by 8:

$$16 \div 8 \quad \text{can be written} \quad 2^4 \div 2^3$$

This equals

$$2^1 \quad \text{or simply} \quad 2$$

using the rules of indices which tell us to subtract the powers 4 and 3 to give the new power, 1.

If we had a look-up table containing powers of 2, it would be straightforward to look up 2^7 and obtain $2^7 = 128$ as the result of finding 16×8.

Notice that by using the powers, we have changed a multiplication problem into one involving addition (the addition of the powers, 4 and 3). Historically, this observation led John Napier (1550-1617) and Henry Briggs (1561-1630) to develop **logarithms** as a way of replacing multiplication with addition, and also division with subtraction.

3. What is a logarithm ?

Consider the expression $16 = 2^4$. Remember that 2 is the base, and 4 is the power. An alternative, yet equivalent, way of writing this expression is $\log_2 16 = 4$. This is stated as 'log to base 2 of 16 equals 4'. We see that the logarithm is the same as the power or index in the original expression. It is the base in the original expression which becomes the base of the logarithm.

The two statements

$$16 = 2^4 \qquad \log_2 16 = 4$$

are equivalent statements. If we write either of them, we are automatically implying the other.

Example

If we write down that $64 = 8^2$ then the equivalent statement using logarithms is $\log_8 64 = 2$.

Example

If we write down that $\log_3 27 = 3$ then the equivalent statement using powers is $3^3 = 27$.

So the two sets of statements, one involving powers and one involving logarithms are equivalent. In the general case we have:

Key Point

if $x = a^n$ then equivalently $\log_a x = n$

Let us develop this a little more.

Because $10 = 10^1$ we can write the equivalent logarithmic form $\log_{10} 10 = 1$.
Similarly, the logarithmic form of the statement $2^1 = 2$ is $\log_2 2 = 1$.

In general, for any base a, $a = a^1$ and so $\log_a a = 1$.

Key Point

$$\log_a a = 1$$

We can see from the Examples above that indices and logarithms are very closely related. In the same way that we have rules or laws of indices, we have **laws of logarithms**. These are developed in the following sections.

4. Exercises

1. Write the following using logarithms instead of powers

a)	$8^2 = 64$	b)	$3^5 = 243$	c)	$2^{10} = 1024$	d)	$5^3 = 125$
e)	$10^6 = 1000000$	f)	$10^{-3} = 0.001$	g)	$3^{-2} = \frac{1}{9}$	h)	$6^0 = 1$
i)	$5^{-1} = \frac{1}{5}$	j)	$\sqrt{49} = 7$	k)	$27^{2/3} = 9$	l)	$32^{-2/5} = \frac{1}{4}$

2. Determine the value of the following logarithms

a)	$\log_3 9$	b)	$\log_2 32$	c)	$\log_5 125$	d)	$\log_{10} 10000$
e)	$\log_4 64$	f)	$\log_{25} 5$	g)	$\log_8 2$	h)	$\log_{81} 3$
i)	$\log_3 \left(\frac{1}{27}\right)$	j)	$\log_7 1$	k)	$\log_8 \left(\frac{1}{8}\right)$	l)	$\log_4 8$
m)	$\log_a a^5$	n)	$\log_c \sqrt{c}$	o)	$\log_s s$	p)	$\log_e \left(\frac{1}{e^3}\right)$

5. The first law of logarithms

Suppose

$$x = a^n \qquad \text{and} \qquad y = a^m$$

then the equivalent logarithmic forms are

$$\log_a x = n \qquad \text{and} \qquad \log_a y = m \qquad (1)$$

Using the first rule of indices

$$xy = a^n \times a^m = a^{n+m}$$

Now the logarithmic form of the statement $xy = a^{n+m}$ is $\log_a xy = n + m$. But $n = \log_a x$ and $m = \log_a y$ from (1) and so putting these results together we have

$$\log_a xy = \log_a x + \log_a y$$

So, if we want to multiply two numbers together and find the logarithm of the result, we can do this by adding together the logarithms of the two numbers. This is the **first law**.

Key Point

$$\log_a xy = \log_a x + \log_a y$$

6. The second law of logarithms

Suppose $x = a^n$, or equivalently $\log_a x = n$. Suppose we raise both sides of $x = a^n$ to the power m:

$$x^m = (a^n)^m$$

Using the rules of indices we can write this as

$$x^m = a^{nm}$$

Thinking of the quantity x^m as a single term, the logarithmic form is

$$\log_a x^m = nm = m \log_a x$$

This is the **second law**. It states that when finding the logarithm of a power of a number, this can be evaluated by multiplying the logarithm of the number by that power.

Key Point

$$\log_a x^m = m \log_a x$$

7. The third law of logarithms

As before, suppose

$$x = a^n \qquad \text{and} \qquad y = a^m$$

with equivalent logarithmic forms

$$\log_a x = n \qquad \text{and} \qquad \log_a y = m \qquad (2)$$

Consider $x \div y$.

$$\begin{aligned} \frac{x}{y} &= a^n \div a^m \\ &= a^{n-m} \end{aligned}$$

using the rules of indices.

In logarithmic form

$$\log_a \frac{x}{y} = n - m$$

which from (2) can be written

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

This is the **third law**.

Key Point

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

8. The logarithm of 1

Recall that any number raised to the power zero is 1: $a^0 = 1$. The logarithmic form of this is

$$\log_a 1 = 0$$

Key Point

$$\log_a 1 = 0$$

The logarithm of 1 in any base is 0.

9. Examples

Example

Suppose we wish to find $\log_2 512$.

This is the same as being asked 'what is 512 expressed as a power of 2 ?'
Now 512 is in fact 2^9 and so $\log_2 512 = 9$.

Example

Suppose we wish to find $\log_8 \frac{1}{64}$.

This is the same as being asked 'what is $\frac{1}{64}$ expressed as a power of 8 ?'

Now $\frac{1}{64}$ can be written 64^{-1}. Noting also that $8^2 = 64$ it follows that

$$\frac{1}{64} = 64^{-1} = (8^2)^{-1} = 8^{-2}$$

using the rules of indices. So $\log_8 \frac{1}{64} = -2$.

Example

Suppose we wish to find $\log_5 25$.

This is the same as being asked 'what is 25 expressed as a power of 5 ?'

Now $5^2 = 25$ and so $\log_5 25 = 2$.

Example

Suppose we wish to find $\log_{25} 5$.

This is the same as being asked 'what is 5 expressed as a power of 25 ?'

We know that 5 is a square root of 25, that is $5 = \sqrt{25}$. So $25^{\frac{1}{2}} = 5$ and so $\log_{25} 5 = \frac{1}{2}$.

Notice from the last two examples that by interchanging the base and the number

$$\log_{25} 5 = \frac{1}{\log_5 25}$$

This is true more generally:

Key Point

$$\log_b a = \frac{1}{\log_a b}$$

To illustrate this again, consider the following example.

Example

Consider $\log_2 8$. We are asking 'what is 8 expressed as a power of 2 ?' We know that $8 = 2^3$ and so $\log_2 8 = 3$.

What about $\log_8 2$? Now we are asking 'what is 2 expressed as a power of 8 ?' Now $2^3 = 8$ and so $2 = \sqrt[3]{8}$ or $8^{1/3}$. So $\log_8 2 = \dfrac{1}{3}$.

We see again

$$\log_8 2 = \frac{1}{\log_2 8}$$

10. Exercises

3 Each of the following expressions can be simplified to $\log N$.
Determine the value of N in each case. We have not explicitly written down the base. You can assume the base is 10, but the results are identical whichever base is used.

a)	$\log 3 + \log 5$	b)	$\log 16 - \log 2$	c)	$3 \log 4$
d)	$2 \log 3 - 3 \log 2$	e)	$\log 236 + \log 1$	f)	$\log 236 - \log 1$
g)	$5 \log 2 + 2 \log 5$	h)	$\log 128 - 7 \log 2$	i)	$\log 2 + \log 3 + \log 4$
j)	$\log 12 - 2 \log 2 + \log 3$	k)	$5 \log 2 + 4 \log 3 - 3 \log 4$	l)	$\log 10 + 2 \log 3 - \log 2$

11. Standard bases

There are two bases which are used much more commonly than any others and deserve special mention. These are

base 10 and base e

Logarithms to base 10, $\log_{10}$, are often written simply as $\log$ without explicitly writing a base down. So if you see an expression like $\log x$ you can assume the base is 10. Your calculator will be pre-programmed to evaluate logarithms to base 10. Look for the button marked log.

The second common base is e. The symbol e is called the **exponential constant** and has a value approximately equal to 2.718. This is a number like π in the sense that it has an infinite decimal expansion. Base e is used because this constant occurs frequently in the mathematical modelling of many physical, biological and economic applications. Logarithms to base e, $\log_{\mathrm{e}}$, are often written simply as $\ln$. If you see an expression like $\ln x$ you can assume the base is e. Such logarithms are also called **Naperian** or **natural** logarithms. Your calculator will be pre-programmed to evaluate logarithms to base e. Look for the button marked ln.

Key Point

Common bases:

$$\log \text{ means } \log_{10} \qquad \ln \text{ means } \log_{\mathrm{e}}$$

where e is the exponential constant.
Useful results:

$$\log 10 = 1, \qquad \ln \mathrm{e} = 1$$

12. Using logarithms to solve equations

We can use logarithms to solve equations where the unknown is in the power.

Suppose we wish to solve the equation $3^x = 5$. We can solve this by taking logarithms of both sides. Whilst logarithms to any base can be used, it is common practice to use base 10, as these are readily available on your calculator. So,

$$\log 3^x = \log 5$$

Now using the laws of logarithms, the left hand side can be re-written to give

$$x \log 3 = \log 5$$

This is more straightforward. The unknown is no longer in the power. Straightaway

$$x = \frac{\log 5}{\log 3}$$

If we wanted, this value can be found from a calculator.

Example

Solve $3^x = 5^{x-2}$. Again, notice that the unknown appears in the power. Take logs of both sides.

$$\log 3^x = \log 5^{x-2}$$

Now use the laws of logarithms.

$$x \log 3 = (x-2)\log 5$$

Notice now that the x we are trying to find is no longer in a power. Multiplying out the brackets

$$x \log 3 = x \log 5 - 2 \log 5$$

Rearrange this equation to get the two terms involving x on one side and the remaining term on the other side.

$$2 \log 5 = x \log 5 - x \log 3$$

Factorise the right-hand side by extracting the common factor of x.

$$\begin{aligned} 2 \log 5 &= x(\log 5 - \log 3) \\ &= x \log \left(\frac{5}{3}\right) \end{aligned}$$

using the laws of logarithms.

And finally

$$x = \frac{2 \log 5}{\log \left(\frac{5}{3}\right)}$$

If we wanted, this value can be found from a calculator.

13. Inverse operations

Suppose we pick a base, 2 say.

Suppose we pick a power, 8 say.

We will now raise the base 2 to the power 8, to give 2^8.

Suppose we now take logarithms to base 2 of 2^8.

We then have

$$\log_2 2^8$$

Using the laws of logarithms we can write this as

$$8 \log_2 2$$

Recall that $\log_a a = 1$, so $\log_2 2 = 1$, and so we have simply 8 again, the number we started with.

So, raising the base 2 to a power, and then finding the logarithm to base 2 of the result are inverse operations.

Let's look at this another way.

Suppose we pick a number, 8 say.

Suppose we find its logarithm to base 2, to evaluate $\log_2 8$.

Suppose we now raise the base 2 to this power: $2^{\log_2 8}$.

Because $8 = 2^3$ we can write this as $2^{\log_2 2^3}$. Using the laws of logarithms this equals $2^{3 \log_2 2}$ which equals 2^3 or 8, since $\log_2 2 = 1$. We see that raising the base 2 to the logarithm of a number to base 2 results in the original number.

So raising a base to a power, and finding the logarithm to that base are inverse operations. Doing one operation, and then following it by the other, we end up where we started.

Example
Suppose we are working in base e. We can pick a number x and evaluate e^x. If we follow this by taking logarithms to base e we obtain

$$\ln \mathrm{e}^x$$

Using the laws of logarithms this equals

$$x \ln \mathrm{e}$$

but $\ln \mathrm{e} = 1$ and so we are left with simply x again. So, raising the base e to a power, and then finding logarithms to base e are inverse operations.

Example
Suppose we are working in base 10. We can pick a number x and evaluate 10^x. If we follow this by taking logarithms to base 10 we obtain

$$\log 10^x$$

Using the laws of logarithms this equals

$$x \log 10$$

but $\log 10 = 1$ and so we are left with simply x again. So, raising the base 10 to a power, and then finding logarithms to base 10 are inverse operations.

Key Point

Similarly,

$$\ln e^x = x, \qquad e^{\ln x} = x$$

$$\log 10^x = x, \qquad 10^{\log x} = x$$

These results will be useful in doing calculus, especially in solving differential equations.

14. Exercises

4 Use logarithms to solve the following equations

a)	$10^x = 5$	b)	$e^x = 8$	c)	$10^x = \frac{1}{2}$	d)	$e^x = 0.1$
e)	$4^x = 12$	f)	$3^x = 2$	g)	$7^x = 1$	h)	$\left(\frac{1}{2}\right)^x = \frac{1}{100}$
i)	$\pi^x = 10$	j)	$e^x = \pi$	k)	$\left(\frac{1}{3}\right)^x = 2$	l)	$10^x = e^{2x-1}$

Answers to Exercises on Logarithms

1.

a)	$\log_8 64 = 2$	b)	$\log_3 243 = 5$	c)	$\log_2 1024 = 10$
d)	$\log_5 125 = 3$	e)	$\log_{10} 1000000 = 6$	f)	$\log_{10} 0.001 = -3$
g)	$\log_3 \left(\frac{1}{9}\right) = -2$	h)	$\log_6 1 = 0$	i)	$\log_5 \left(\frac{1}{5}\right) = -1$
j)	$\log_{49} 7 = \frac{1}{2}$	k)	$\log_{27} 9 = \frac{2}{3}$	l)	$\log_{32} \left(\frac{1}{4}\right) = -\frac{2}{5}$

2.

a)	2	b)	5	c)	3	d)	4
e)	3	f)	$\frac{1}{2}$	g)	$\frac{1}{3}$	h)	$\frac{1}{4}$
i)	-3	j)	0	k)	-1	l)	$\frac{3}{2}$
m)	5	n)	$\frac{1}{2}$	o)	1	p)	-3

3.

a)	15	b)	8	c)	64	d)	$\frac{9}{8}$
e)	236	f)	236	g)	800	h)	1
i)	24	j)	9	k)	$\frac{2592}{64} = \frac{81}{2}$	l)	45

4. All answers are correct to 3 decimal places

a)	0.699	b)	2.079	c)	-0.301	d)	-2.303
e)	1.792	f)	0.631	g)	0	h)	6.644
i)	2.011	j)	1.145	k)	-0.631	l)	-3.305

What is a logarithm ?

mc-logs1-2009-1

Logarithms appear in many applications and familiarity with them is essential. They are used to write expressions involving powers in different forms.

Logarithms

Study the statement

$$100 = 10^2$$

In this statement we say that 10 is the **base** and 2 is the **power** or **index**. **Logarithms** provide an alternative way of writing a statement such as this. We rewrite it as

$$\log_{10} 100 = 2$$

This is read as 'log to the base 10 of 100 is 2'. These alternative forms are shown in Figure 1.

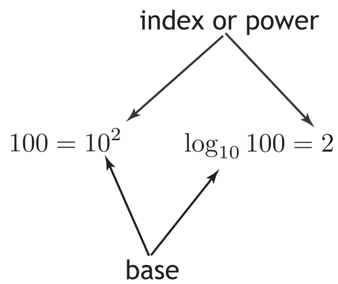


Figure 1. Note the positions of the different quantities in these two alternative forms.

As another example, since

$$2^5 = 32$$

we can write

$$\log_2 32 = 5$$

Here the base is 2 and the power is 5. We read this as 'log to the base 2 of 32 is 5'.

More generally,

> if $a = b^c$, then $\log_b a = c$

Exercises

1. Rewrite the following expressions in logarithm form. Do not try to use a calculator.

(a) $3^2 = 9$ (b) $5^4 = 625$ (c) $10^3 = 1000$ (d) $10^{-2} = 0.01$

(e) $10^1 = 10$ (f) $2^1 = 2$ (g) $\mathrm{e}^1 = \mathrm{e}$ (h) $8^1 = 8.$

2. Rewrite the following expressions in an equivalent form without using logarithms. Do not use a calculator.

(a) $\log_2 256 = 8$ (b) $\log_{10} 10000 = 4$ (c) $\log_4 64 = 3$ (d) $\log_{10} 0.1 = -1$

(e) $\log_3 3 = 1$ (f) $\log_9 9 = 1$ (g) $\log_8 1 = 0$ (h) $\log_2 1 = 0.$

Using a calculator to find logarithms

The only restriction that is placed on the value of the base is that it is a positive real number excluding the number 1. In practice logarithms are calculated using only a few common bases. Most frequently you will meet bases 10 and e. The letter e stands for the number 2.718... and is used because it is found to occur in the mathematical description of many physical phenomena. The number e is called the **exponential constant**. Your calculator will be able to calculate logarithms to bases 10 and e. Usually the 'log' button is used for base 10, and the 'ln' button is used for base e. ('ln' stands for 'natural logarithm'). Check that you can use your calculator correctly by verifying that

$$\log_{10} 73 = 1.8633 \text{ (to 4 decimal places)}$$

and

$$\log_{\mathrm{e}} 5.64 = 1.7299 \text{ (to 4 decimal places)}$$

You may also like to verify the alternative forms

$$10^{1.8633} = 73 \qquad \text{and } \mathrm{e}^{1.7299} = 5.64$$

Occasionally we need to find logarithms to other bases. For example, logarithms to the base 2 are used in communications engineering and information technology. Your calculator can still be used but we need to apply a formula for changing the base. This is dealt with on the leaflet *Logs - changing the base*.

Answers

1. (a) $\log_3 9 = 2$ (b) $\log_5 625 = 4$ (c) $\log_{10} 1000 = 3$ (d) $\log_{10} 0.01 = -2$

   (e) $\log_{10} 10 = 1$ (f) $\log_2 2 = 1$ (g) $\log_{\mathrm{e}} \mathrm{e} = 1$ (h) $\log_8 8 = 1$

2. (a) $2^8 = 256$ (b) $10^4 = 10000$ (c) $4^3 = 64$ (d) $10^{-1} = 0.1$

   (e) $3^1 = 3$ (f) $9^1 = 9$ (g) $8^0 = 1$ (h) $2^0 = 1$

Engineering Maths First Aid Kit

3.8

Solving equations involving logarithms and exponentials

Introduction

It is often necessary to solve an equation in which the unknown occurs as a power, or exponent. For example, you may need to find the value of x which satisfies $2^x = 32$. Very often the base will be the exponential constant e, as in the equation $\mathrm{e}^x = 20$. To understand what follows you must be familiar with the exponential constant. See leaflet *3.4 The exponential constant* if necessary.

You will also come across equations involving logarithms. For example you may need to find the value of x which satisfies $\log_{10} x = 34$. You will need to understand what is meant by a logarithm, and the laws of logarithms (leaflets *2.19 What is a logarithm ?* and *2.20 The laws of logarithms*). On this leaflet we explain how such equations can be solved.

1. Revision of logarithms

Logarithms provide an alternative way of writing expressions involving powers. If

$$a = b^c \qquad \text{then} \qquad \log_b a = c$$

For example: $100 = 10^2$ can be written as $\log_{10} 100 = 2$.

Similarly, $\mathrm{e}^3 = 20.086$ can be written as $\log_\mathrm{e} 20.086 = 3$.

The third law of logarithms states that, for logarithms of any base,

$$\log A^n = n \log A$$

For example, we can write $\log_{10} 5^2$ as $2 \log_{10} 5$, and $\log_\mathrm{e} 7^3$ as $3 \log_\mathrm{e} 7$.

2. Solving equations involving powers

Example
Solve the equation $\mathrm{e}^x = 14$.

Solution
Writing $\mathrm{e}^x = 14$ in its alternative form using logarithms we obtain $x = \log_\mathrm{e} 14$, which can be evaluated directly using a calculator to give 2.639.

Example
Solve the equation $e^{3x} = 14$.

Solution
Writing $e^{3x} = 14$ in its alternative form using logarithms we obtain $3x = \log_e 14 = 2.639$. Hence $x = \frac{2.639}{3} = 0.880$.

To solve an equation of the form $2^x = 32$ it is necessary to take the logarithm of both sides of the equation. This is referred to as 'taking logs'. Usually we use logarithms to base 10 or base e because values of these logarithms can be obtained using a scientific calculator.

Starting with $2^x = 32$, then taking logs produces $\log_{10} 2^x = \log_{10} 32$. Using the third law of logarithms, we can rewrite the left-hand side to give $x \log_{10} 2 = \log_{10} 32$. Dividing both sides by $\log_{10} 2$ gives

$$x = \frac{\log_{10} 32}{\log_{10} 2}$$

The right-hand side can now be evaluated using a calculator in order to find x:

$$x = \frac{\log_{10} 32}{\log_{10} 2} = \frac{1.5051}{0.3010} = 5$$

Hence $2^5 = 32$. Note that this answer can be checked by substitution into the original equation.

3. Solving equations involving logarithms

Example
Solve the equation $\log_{10} x = 0.98$

Solution
Rewriting the equation in its alternative form using powers gives $10^{0.98} = x$. A calculator can be used to evaluate $10^{0.98}$ to give $x = 9.550$.

Example
Solve the equation $\log_e 5x = 1.7$

Solution
Rewriting the equation in its alternative form using powers gives $e^{1.7} = 5x$. A calculator can be used to evaluate $e^{1.7}$ to give $5x = 5.4739$ so that $x = 1.095$ to 3dp.

Exercises
1. Solve each of the following equations to find x.

a) $3^x = 15$, b) $e^x = 15$, c) $3^{2x} = 9$, d) $e^{5x-1} = 17$, e) $10^{3x} = 4$.

2. Solve the equations a) $\log_e 2x = 1.36$, b) $\log_{10} 5x = 2$, c) $\log_{10}(5x + 3) = 1.2$.

Answers
1. a) 2.465, b) 2.708, c) 1, d) 0.767, e) 0.201.

2. a) 1.948, (3dp). b) 20, c) 2.570 (3dp).

Solving Logarithmic Equations

The properties of logarithms can be used to solve logarithmic equations.

For any positive numbers m, n, and b where $b \neq 1$:

Product Property $\log_b mn = \log_b m + \log_b n$

Quotient Property $\log_b \frac{m}{n} = \log_b m - \log_b n$

Power Property $\log_b m^n = n \log_b m$

Example: What is the solution of $\log 3 + \log 2x = 2$?

$\log 3 + \log 2x = 2$

$\log (6x) = 2$ Product Property

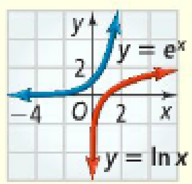
$10^{\log 6x} = 10^2$ Exponential Form

$6x = 100$

$x = \frac{50}{3}$

Natural Logarithms

Natural logarithms (ln) are logarithms to the base e. They follow the same rules and properties as logarithms to other bases. The functions $y = e^x$ and $y = \ln x$ are inverse functions.



Example: What are the solutions of $\ln 3x^2 - 2 \ln 3 = 2$?

$\ln 3x^2 - 2 \ln 3 = 2$

$\ln 3x^2 - \ln 3^2 = 2$ Power Property

$\ln \frac{x^2}{3} = 2$ Quotient Property

$\frac{x^2}{3} = e^2$ Exponential Form

$x^2 \approx 22.1672$

$x \approx \pm 4.708$

Common Errors When Solving Logarithmic Equations

Quotient Property Students sometimes mistakenly apply the Quotient Property in the reverse order. Point out that the log of the numerator is the first term in the subtraction expression.

Exponential Form Errors occur when students write logarithmic equations in exponential form. Encourage them to write steps as needed. For example, in the natural logarithm example above, write $e^{\ln\frac{x^2}{3}} = e^2$ to get the equation in exponential form. Because natural log and e are inverses, the left side of the equation becomes $\frac{x^2}{3}$.

Solving Exponential Equations

Solve Algebraically

Exponential equations can be solved by taking logarithms of each side. Although logarithms to any base can be used, common logs and natural logs are generally used.

Example: Solve $13^{n+10} = 80$.

$13^{n+10} = 80$

$\log 13^{n+10} = \log 80$

$(n + 10)(\log 13) = \log 80$ Power Property

$n + 10 = \frac{\log 80}{\log 13}$

$n \approx -8.9216$

An equation need not contain e to use the natural log.

Example: What is the solution of $5^x = 30$?

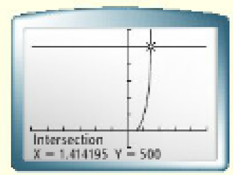
$5^x = 30$	$5^x = 30$
$x \ln 5 = \ln 30$	$x \log 5 = \log 30$
$x = \frac{\ln 30}{\ln 5}$	$x = \frac{\log 30}{\log 5}$
$x \approx 2.113$	$x \approx 2.113$

Solve Graphically

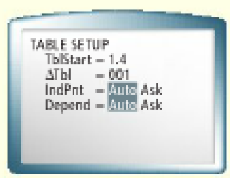
Both exponential and logarithmic equations can be solved using a graph or table.

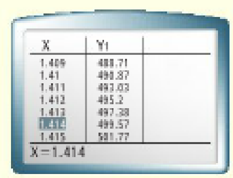
Example: Solve $3^{4x} = 500$.

Enter $3^{(4x)}$ in Y_1 and 500 in Y_2. Find the intersection.



Solve using the table.





Common Errors When Solving Exponential Equations

Same Bases: A method for solving exponential equations involves equating the exponents if the bases are the same. Students may try to equate the exponents when the bases are not the same. Different bases require the use of logarithms. Have students check their answers to be sure their solution is correct.

Logs of negative numbers: Students can often be confused as to the restrictions for the variables and logarithmic equations. Point out that a positive number to any power can never been negative. Therefore, it is not possible to find a logarithm of a negative number. For example, there is no solution to the equation $10^x = -100$.